GOLDEN GASKETS: VARIATIONS ON THE SIERPIŃSKI SIEVE

DAVE BROOMHEAD, JAMES MONTALDI, AND NIKITA SIDOROV

ABSTRACT. We consider the iterated function systems (IFSs) that consist of three general similitudes in the plane with centres at three non-collinear points, and with a common contraction factor $\lambda \in (0,1)$.

As is well known, for $\lambda = 1/2$ the invariant set, $\mathcal{S}_\lambda$, is a fractal called the Sierpiński sieve, and for $\lambda < 1/2$ it is also a fractal. Our goal is to study $\mathcal{S}_\lambda$ for this IFS for $1/2 < \lambda < 2/3$, i.e., when there are "overlaps" in $\mathcal{S}_\lambda$ as well as "holes". In this introductory paper we show that despite the overlaps (i.e., the Open Set Condition breaking down completely), the attractor can still be a totally self-similar fractal, although this happens only for a very special family of algebraic λ's (so-called "multinacci numbers"). We evaluate $\dim_H(\mathcal{S}_\lambda)$ for these special values by showing that $\mathcal{S}_\lambda$ is essentially the attractor for an infinite IFS which does satisfy the Open Set Condition. We also show that the set of points in the attractor with a unique "address" is self-similar, and compute its dimension.

For "non-multinacci" values of λ we show that if λ is close to $2/3$, then $\mathcal{S}_\lambda$ has a nonempty interior and that if $\lambda < 1/\sqrt{3}$ then $\mathcal{S}_\lambda$ has zero Lebesgue measure. Finally we discuss higher-dimensional analogues of the model in question.

INTRODUCTION AND SUMMARY

Iterated function systems are one of the most common tools for constructing fractals. Usually, however, a very special class of IFSs is considered for this purpose, namely, those which satisfy the *Open Set Condition* (OSC)—see below. We present—apparently for the first time—a family of simple and natural examples of fractals that originate from IFSs for which the OSC is violated; that is, for which substantial overlaps occur.

We consider a family of iterated function systems (IFSs) defined by taking three planar similitudes $f_i(\mathbf{x}) = \lambda\mathbf{x} + (1-\lambda)\mathbf{p}_i$ ($i = 0,1,2$), where the scaling factor $\lambda \in (0,1)$ and the centres $\mathbf{p}_i$ are three non-collinear points in $\mathbb{R}^2$. Without loss of generality we take the centres to be at the vertices of an equilateral triangle Δ (see Section 7). The resulting IFS has a unique compact invariant set $\mathcal{S}_\lambda$ (depending on λ); by definition $\mathcal{S}_\lambda$ satisfies

$$\mathcal{S}_\lambda = \bigcup_{i=0}^{2} f_i(\mathcal{S}_\lambda).$$

More conveniently, $\mathcal{S}_\lambda$ can be found (or rather approximated) inductively by iterating the f_j. Let

$$\Delta_n = \bigcup_{\boldsymbol{\varepsilon}\in\Sigma^n} f_{\boldsymbol{\varepsilon}}(\Delta),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{n-1}) \in \Sigma^n$, and $\Sigma = \{0,1,2\}$, and

$$f_{\boldsymbol{\varepsilon}} = f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}.$$

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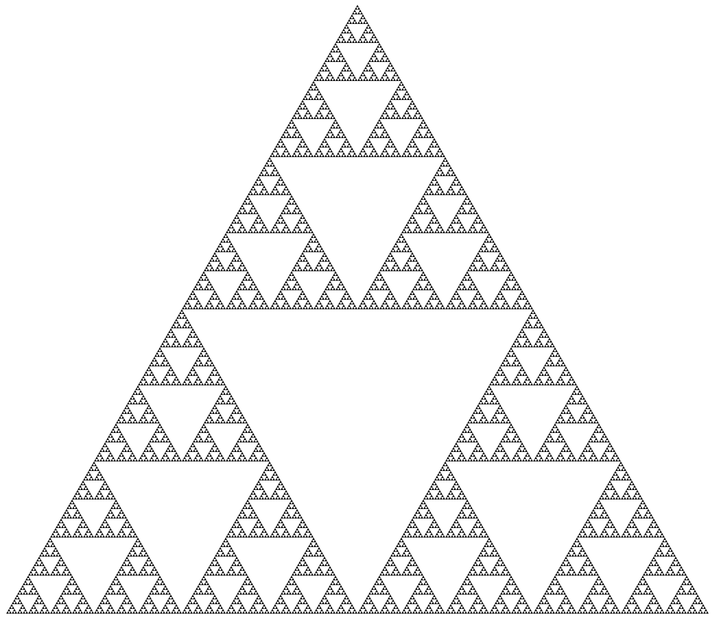


FIGURE 1. The Sierpiński Sieve.

Since $f_i(\Delta) \subset \Delta$ it follows that $\Delta_{n+1} \subset \Delta_n$ and then

$$\mathcal{S}_\lambda = \lim_{n\to\infty} \Delta_n = \bigcap_{n=1}^{\infty} \Delta_n.$$

In fact all our figures are produced (using Mathematica) by drawing Δ_n for n suitably large, typically between 7 and 10.

For $\lambda \leq 1/2$ the images of the three similarities are essentially disjoint (more precisely, the similarities satisfy the open set condition (OSC)), which makes the invariant set relatively straightforward to analyse. For $\lambda = 1/2$ the invariant set is the famous Sierpiński sieve (or triangle or gasket)—see Figure 1, and for $\lambda \leq 1/2$ the invariant set is a self-similar fractal of dimension $\log 3/(-\log\lambda)$. On the other hand, if $\lambda \geq 2/3$ the union of the three images coincides with the original triangle[1] Δ, so that $\mathcal{S}_\lambda = \Delta$.

In this paper we begin a systematic study of the IFS for the remaining values of λ, namely for $\lambda \in (1/2,\, 2/3)$. In this region, the three images have significant overlaps, and the IFS does not satisfy the open set condition, which makes it much harder to study properties of the invariant set. For example, it is not known precisely for which values of λ it has positive Lebesgue measure. We do, however, obtain partial results: for $\lambda < 1/\sqrt{3} \approx 0.577$ the invariant set has zero Lebesgue measure (Proposition 4.3), while for $\lambda \geq \lambda_* \approx 0.648$ it has non-zero Lebesgue measure (Proposition 2.7). We also show (Proposition 4.1) that the Lebesgue measure vanishes for the specific value $\lambda = (\sqrt{5}-1)/2 \approx 0.618$.

The main result of this paper is that there is a countable family of values of λ in the interval $(1/2, 2/3)$—the so-called *multinacci numbers* ω_m—for which the invariant set $\mathcal{S}_\lambda$ is totally self-similar (Definition 1.2). We call the resulting invariant sets *Golden Gaskets*, and the first three golden gaskets are shown in Figures 2, 3 and 4 respectively. For these values of λ we are able to compute the Hausdorff dimension of $\mathcal{S}_\lambda$ (Theorem 4.4). The multinacci numbers are defined as follows. For each

[1]By "triangle" we always mean the convex hull of three points, not just the boundary.

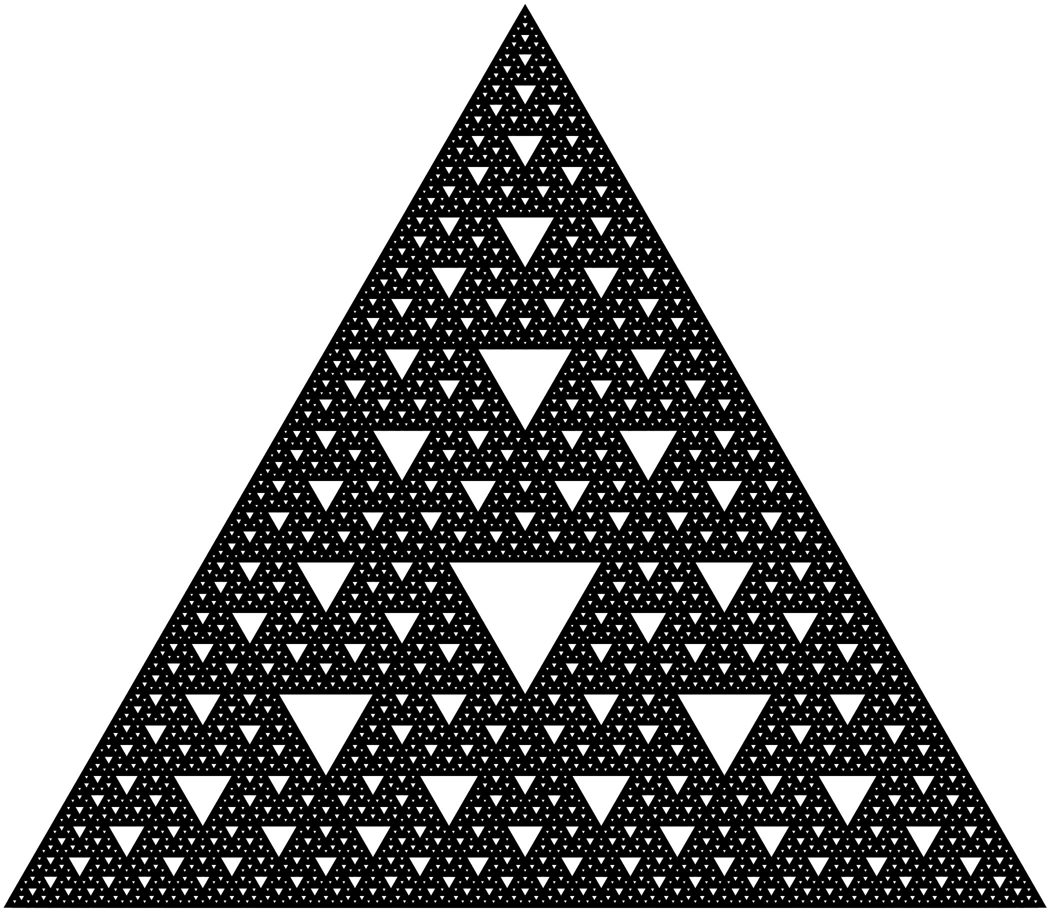


FIGURE 2. *The* Golden Gasket $\mathcal{S}_{\omega_2}$

$m \geq 2$ the multinacci number ω_m is defined to be the positive solution of the equation

$$x^m + x^{m-1} + \cdots + x = 1.$$

The first multinacci number is the golden ratio $\omega_2 = (\sqrt{5}-1)/2 \approx 0.618$, and the second is $\omega_3 \approx 0.544$. It is easy to see that as m increases, so ω_m decreases monotonically, converging to 1/2.

The key property responsible for the invariant set being totally self-similar for the multinacci numbers is that for these values of λ the overlap $f_i(\Delta) \cap f_j(\Delta)$ is an image of Δ, namely it coincides with $f_i f_j^m(\Delta)$. On the other hand, we also show that if $\lambda \in (1/2, 2/3)$ is not a multinacci number then the invariant set is not totally self-similar (Theorem 5.3).

The paper is organized as follows. In Section 1 we define the IFS, and introduce the barycentric coordinates we use for all calculations. In Section 2 we describe the distribution of holes in the invariant set, and deduce that for $\lambda \geq \lambda_* \approx 0.6478$ the invariant set has nonempty interior (Proposition 2.7).

In Section 3 we describe explicitly the new family of *golden gaskets*. The main result is that for these values of λ the invariant set is totally self-similar (Theorem 3.3). In Section 4 we give several results on the Lebesgue measure and the Hausdorff dimension of the invariant set, as described above. The main result of Section 5 is that if λ is not multinacci, then the invariant set is not totally self

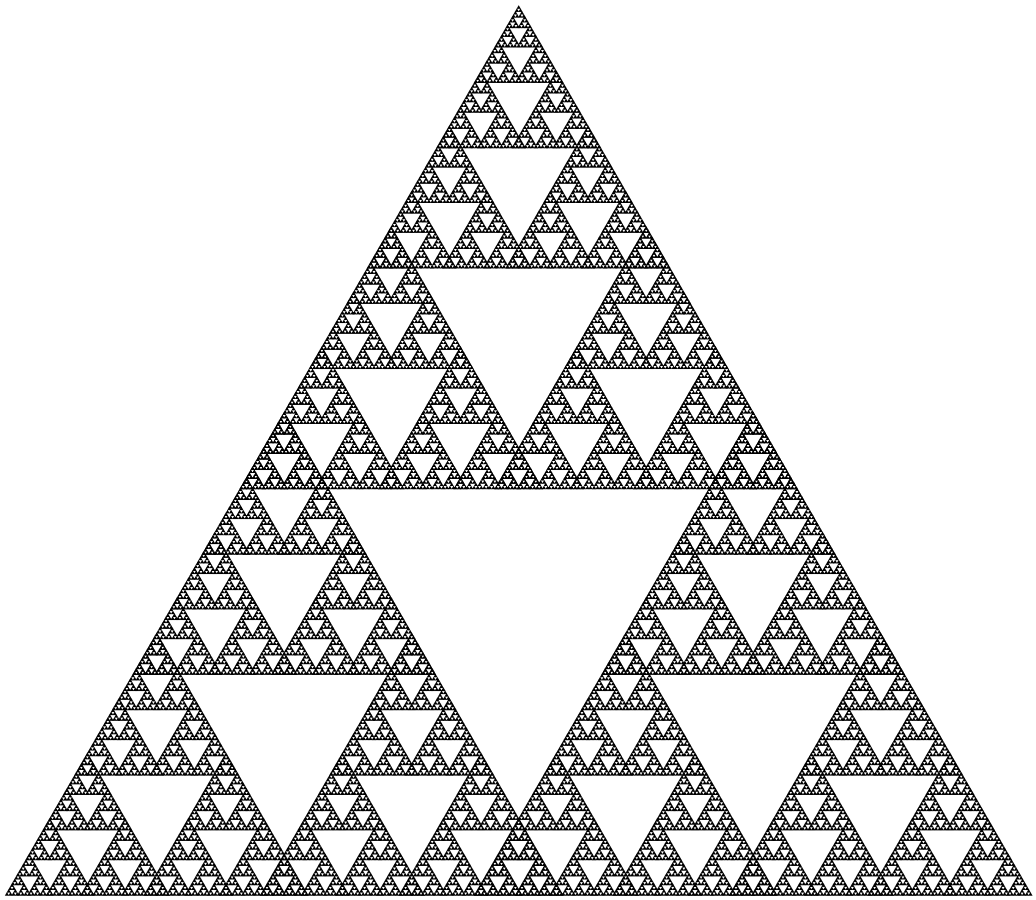


FIGURE 3. The invariant set $\mathcal{S}_{\omega_3}$

similar. In Corollary 5.5 we show how this theorem can be used to prove a result in number theory—an upper bound for the "separation constant" that is already known but our proof is very different, and simpler.

There are two ways to generalize this model: one is to introduce more similitudes in the plane, and the other is to pass into higher dimensions, but remaining with simplices (generalizing the equilateral triangle to higher dimensions). The first is very much harder than the second, and in Section 6 we consider the second by way of a very brief discussion of the "golden sponges" and a list of a few results that can be obtained by the same arguments as for the planar case. Finally, in Section 7 we end with a few remarks and open questions.

The appendix contains a detailed proof of the dimensions formula for the golden gaskets. In doing this, we need to consider points of the invariant set as determined by a symbolic address: to $\boldsymbol{\varepsilon} \in \Sigma^\infty$ one associates $\mathbf{x}_{\boldsymbol{\varepsilon}} \in \mathcal{S}_\lambda$ by $\mathbf{x}_{\boldsymbol{\varepsilon}} = \lim_n f_{\varepsilon_0} \dots f_{\varepsilon_n}(x_0)$ (independently of x_0). The *set of uniqueness* $\mathcal{U}_\lambda$ consists of those points in the invariant set that have only one symbolic address in Σ^∞. For $\lambda = \omega_m$ a multinacci number, we show $\mathcal{U}_{\omega_m}$ to be a self-similar set, and compute its Hausdorff dimension in Theorem A.4. We also show that "almost every" point of $\mathcal{S}_{\omega_m}$ (in the sense of prevailing dimension) has a continuum of different "addresses" (Proposition A.5).

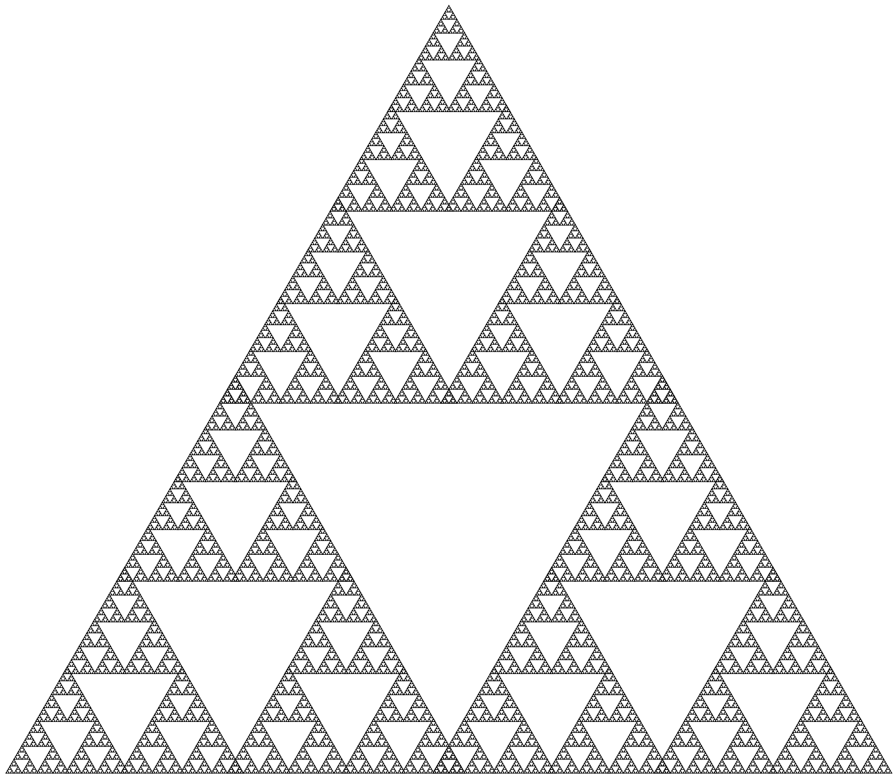


FIGURE 4. The invariant set $\mathcal{S}_{\omega_4}$. Notice the close resemblance to the Sierpiński sieve in Figure 1.

This is apparently the first paper, where a family of IFSs in $\mathbb{R}^2$ with both holes and overlaps is considered in detail. In $\mathbb{R}$, however, there has been an attempt to do this, namely, the famous "0,1,3"-problem. More precisely, the maps for that model are as follows: $g_j(x) = \lambda x + (1-\lambda)j$, where $x \in \mathbb{R}$ and $j \in \{0,1,3\}$. Unfortunately, the problem of describing the invariant set for this IFS with $\lambda \in (1/3, 2/5)$ (which is exactly the "interesting" region) has proved to be very complicated, and only partial results have been obtained so far—see [11, 14] for more detail.

1. THE ITERATED FUNCTION SYSTEM

Our set-up is as follows. Let $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}_2$ be the vertices of the equilateral triangle Δ:

$$\mathbf{p}_k = \frac{2}{3}(\cos(2\pi k/3),\ \sin(2\pi k/3)),\ \ k = 0,1,2$$

(this choice of the scaling will become clear later). Let f_0, f_1, f_2 be three contractions defined as

$$f_i(\mathbf{x}) = \lambda\mathbf{x} + (1-\lambda)\mathbf{p}_i, \quad i = 0,1,2. \tag{1.1}$$

Under composition, these functions generate an *iterated function system* (IFS)[2].

[2]Often, in the literature, the term "IFS" means a random functions system endowed with probabilities. Our model however will be purely topological.

The invariant set (or attractor) of this IFS is defined to be the unique non-empty compact set $\mathcal{S}_\lambda$ satisfying

$$\mathcal{S}_\lambda = \bigcup_{i=0}^{2} f_i(\mathcal{S}_\lambda).$$

An iterative procedure exists as follows (see, e.g., [7]): let $\Delta_0 := \Delta$ and

$$\Delta_n := \bigcup_{i=0}^{2} f_i(\Delta_{n-1}), \quad n \geq 1. \tag{1.2}$$

The invariant set is then:

$$\mathcal{S}_\lambda = \bigcap_{n=0}^{\infty} \Delta_n = \lim_{n\to+\infty} \Delta_n,$$

where the limit is taken in the Hausdorff metric.

From here on $\Sigma := \{0,1,2\}$, $\boldsymbol{\varepsilon}$ denotes $(\varepsilon_0 \dots \varepsilon_{n-1})$ (for some n) and

$$f_{\boldsymbol{\varepsilon}} := f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}.$$

As is easy to see by induction,

$$\Delta_n = \bigcup_{\boldsymbol{\varepsilon}\in\Sigma^n} f_{\boldsymbol{\varepsilon}}(\Delta),$$

whence $\Delta_n \subset \Delta_{n-1}$.

A well studied case is $\lambda = \frac{1}{2}$, which leads to the *Sierpiński sieve* (or *Sierpiński gasket* or *triangle*) $\mathcal{S} := \mathcal{S}_{1/2}$—see Figure 1. Figures 2, 3 and 4 show the first three of the new sequence of fractals, for $\lambda = \omega_2, \omega_3$ and ω_4 respectively (the first three multinacci numbers).

Definition 1.1. Recall that the Open Set Condition (OSC) is defined as follows: let O be the interior of Δ; then $\bigcup_i f_i(O) \subset O$, the union being disjoint.

Note that for $\lambda = 1/2$ the intersections $f_i(\Delta) \cap f_j(\Delta)$ $(i,j = 0,1,2,\ i \neq j)$ are two-point sets, i.e., by definition, this IFS satisfies the Open Set Condition. We would like to emphasize one more important property of the Sierpiński sieve. Looking at Figure 1, one immediately sees that each smaller triangle has the same structure of holes as the big one. In other words,

$$f_{\boldsymbol{\varepsilon}}(\mathcal{S}) = f_{\boldsymbol{\varepsilon}}(\Delta) \cap \mathcal{S} \quad \text{for any } \boldsymbol{\varepsilon} \in \Sigma^n \text{ and any } n. \tag{1.3}$$

Definition 1.2. We call any set $\mathcal{S}$ that satisfies (1.3), *totally self-similar*.

Total self-similarity in the case of the Sierpiński sieve implies, in particular, its holes being well structured: the n^{th} "layer" of holes—i.e., $\Delta_n \setminus \Delta_{n+1}$—contains 3^n holes (the central hole being layer zero), and each of these is surrounded (at a distance depending on n only) by exactly three holes of the $(n+1)^{\text{th}}$ layer, each smaller in size by the factor λ ($= 1/2$ in this case). Later we will see that only very special values of λ yield this property of $\mathcal{S}_\lambda$.

If $\lambda < 1/2$, we have the OSC as well (the intersections $f_i(\Delta) \cap f_j(\Delta),\ i \neq j$ are clearly empty). However, if $\lambda \in (1/2, 1)$, then $f_i(\Delta) \cap f_j(\Delta)$ is always a triangle, which means that the OSC is not satisfied. This changes the invariant set dramatically. Our goal is to show that there exists a countable family of parameters between $1/2$ and 1 which, despite the lack of the OSC, provide total self-similarity of $\mathcal{S}_\lambda$ and, conversely, that for all other λ's there cannot be total self-similarity.

For technical purposes we introduce a system of coordinates in Δ that is more convenient than the usual Cartesian coordinates. Namely, we identify each point $\mathbf{x} \in \Delta$ with a triple (x, y, z), where

$$x = \text{dist}\,(\mathbf{x}, [\mathbf{p}_1, \mathbf{p}_2]),\ y = \text{dist}\,(\mathbf{x}, [\mathbf{p}_0, \mathbf{p}_2]),\ z = \text{dist}\,(\mathbf{x}, [\mathbf{p}_0, \mathbf{p}_1]),$$

where $[\mathbf{p}_i, \mathbf{p}_j]$ is the edge containing $\mathbf{p}_i$ and $\mathbf{p}_j$. As used to be well known from high-school geometry, $x+y+z$ equals the tripled radius of the inscribed circle, i.e., in our case, 1 (this is why we have chosen the radius of the circumcircle for our triangle to be equal to $2/3$). These are usually called *barycentric coordinates*.

Lemma 1.3. *In barycentric coordinates f_0, f_1, f_2 act as linear maps. More precisely,*

$$f_0 = \begin{pmatrix} 1 & 1-\lambda & 1-\lambda \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \ f_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 1-\lambda & 1 & 1-\lambda \\ 0 & 0 & \lambda \end{pmatrix}, \ f_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 1-\lambda & 1-\lambda & 1 \end{pmatrix}.$$

Proof. The fact that the f_i are linear in barycentric coordinates is a trivial consequence of the f_i being affine. Let us show that the matrix for f_0, say, has the given form (the proof for f_1, f_2 is exactly the same).

Note first that since any vector (x, y, z) is stochastic, so must be $f_0 = (a_{ij})_{i,j=1}^3$, i.e., $a_{ij} \geq 0$, $\sum_j a_{ij} = 1$ for any $i = 1, 2, 3$. Now, f_0 as a map acts as a contraction in the direction of $\mathbf{p}_0$, which implies that on y or z it acts simply by multiplying it by λ. Therefore, $a_{21} = 0, a_{22} = \lambda, a_{23} = 0, a_{31} = a_{32} = 0, a_{33} = \lambda$, and by the stochasticity of f_0, $a_{11} = 1, a_{12} = a_{13} = 1 - \lambda$. $\square$

From here on by a *hole* we mean a connected component in $\Delta \setminus \mathcal{S}_\lambda$. First of all, we show that if $\lambda \geq 2/3$, then there are no holes at all:

Lemma 1.4. *If $\lambda \in [2/3, 1)$, then $\mathcal{S}_\lambda = \Delta$.*

Proof. It suffices to show that $\bigcup_i f_i(\Delta) = \Delta$. In barycentric coordinates, $f_0(\Delta) = \{x \geq 1 - \lambda\}, f_1(\Delta) = \{y \geq 1 - \lambda\}, f_2(\Delta) = \{z \geq 1 - \lambda\}$. For (x, y, z) to lie in the hole, therefore, the conditions $x < 1 - \lambda$, $y < 1 - \lambda$ and $z < 1 - \lambda$ must be satisfied simultaneously. Since $\lambda \geq 2/3$ and $x + y + z = 1$, this is impossible. $\square$

2. Structure of the holes

Thus, the "interesting" region is $\lambda \in (1/2, 2/3)$. Let H_0 denote the *central hole*, i.e., $H_0 = \Delta \setminus \Delta_1$; it is an "inverted" equilateral triangle.

Lemma 2.1. *Each hole is a subset of $\bigcup_{\boldsymbol{\varepsilon} \in \Sigma^n} f_{\boldsymbol{\varepsilon}}(H_0)$ for some $n \geq 1$.*

Proof. If $\mathbf{x}$ is in a hole, then there exists $n \geq 1$ such that $\mathbf{x} \in \Delta_n \setminus \Delta_{n+1}$. Now our claim follows from

$$\Delta_n \setminus \Delta_{n+1} = \bigcup_{\boldsymbol{\varepsilon}} f_{\boldsymbol{\varepsilon}}(\Delta) \setminus \bigcup_{\boldsymbol{\varepsilon}} f_{\boldsymbol{\varepsilon}}(\Delta_1) = \bigcup_{\boldsymbol{\varepsilon}} f_{\boldsymbol{\varepsilon}}(\Delta \setminus \Delta_1) = \bigcup_{\boldsymbol{\varepsilon}} f_{\boldsymbol{\varepsilon}}(H_0).$$

$\square$

Remark 2.2. Note that although the $f_{\boldsymbol{\varepsilon}}(H_0)$ may not be disjoint, any hole is in fact an inverted triangle and a subset of just one of the $f_{\boldsymbol{\varepsilon}}(H_0)$. We leave this claim without proof, as it is not needed.

Let us now derive the formula for any finite combination of f_i. Put

$$a_k = \begin{cases} 1, & \varepsilon_k = 0 \\ 0, & \text{otherwise} \end{cases}, \ b_k = \begin{cases} 1, & \varepsilon_k = 1 \\ 0, & \text{otherwise} \end{cases}, \ c_k = \begin{cases} 1, & \varepsilon_k = 2 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, a_k, b_k, c_k are 0's and 1's and $a_k + b_k + c_k = 1$.

Lemma 2.3. *Let $\varepsilon_k \in \Sigma$ for $k = 0, 1, \dots, n$. Then*

$$f_{\boldsymbol{\varepsilon}} = \begin{pmatrix} (1-\lambda)\sum_0^{n-1} a_k\lambda^k + \lambda^n & (1-\lambda)\sum_0^{n-1} a_k\lambda^k & (1-\lambda)\sum_0^{n-1} a_k\lambda^k \\ (1-\lambda)\sum_0^{n-1} b_k\lambda^k & (1-\lambda)\sum_0^{n-1} b_k\lambda^k + \lambda^n & (1-\lambda)\sum_0^{n-1} b_k\lambda^k \\ (1-\lambda)\sum_0^{n-1} c_k\lambda^k & (1-\lambda)\sum_0^{n-1} c_k\lambda^k & (1-\lambda)\sum_0^{n-1} c_k\lambda^k + \lambda^n \end{pmatrix}.$$

Proof. Induction: for $n = 1$ this is obviously true; assume that the formula is valid for some n and verify its validity for $n+1$. Within this proof, we write

$$p_n = (1-\lambda)\sum_0^{n-1} a_k\lambda^k, \quad q_n = (1-\lambda)\sum_0^{n-1} b_k\lambda^k, r_n = (1-\lambda)\sum_0^{n-1} c_k\lambda^k.$$

Then by our assumption,

$$\begin{aligned} f_{\boldsymbol{\varepsilon}} f_0 &= \begin{pmatrix} p_n + \lambda^n & p_n & p_n \\ q_n & q_n + \lambda^n & q_n \\ r_n & r_n & r_n + \lambda^n \end{pmatrix} \begin{pmatrix} 1 & 1-\lambda & 1-\lambda \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} p_n + \lambda^n & p_n + (1-\lambda)\lambda^n & p_n + (1-\lambda)\lambda^n \\ q_n & q_n + \lambda^{n+1} & q_n \\ r_n & r_n & r_n + \lambda^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} p_{n+1} + \lambda^{n+1} & p_{n+1} & p_{n+1} \\ q_{n+1} & q_{n+1} + \lambda^{n+1} & q_{n+1} \\ r_{n+1} & r_{n+1} & r_{n+1} + \lambda^{n+1} \end{pmatrix}, \end{aligned}$$

as $p_{n+1} = \sum_0^n a_k\lambda^k = (1-\lambda)\left(\sum_0^{n-1} a_k\lambda^k + \lambda^n\right)$, whence $p_n + \lambda^n = p_{n+1} + \lambda^{n+1}$. For q_n and r_n we have $q_{n+1} = q_n,\ r_{n+1} = r_n$. Multiplication by f_1 and f_2 is considered in the same way. □

Corollary 2.4. *We have*

$$f_{\boldsymbol{\varepsilon}}(\Delta) = \begin{cases} x & \geq (1-\lambda)\sum_{k=0}^{n-1} a_k\lambda^k, \\ y & \geq (1-\lambda)\sum_{k=0}^{n-1} b_k\lambda^k, \\ z & \geq (1-\lambda)\sum_{k=0}^{n-1} c_k\lambda^k. \end{cases}$$

Proof. The set $f_{\boldsymbol{\varepsilon}}(\Delta)$ is the triangle with the vertices $f_{\boldsymbol{\varepsilon}}(\mathbf{p}_0), f_{\boldsymbol{\varepsilon}}(\mathbf{p}_1)$ and $f_{\boldsymbol{\varepsilon}}(\mathbf{p}_2)$. By definition, in this triangle x is greater than or equal to the joint first coordinate of $f_{\boldsymbol{\varepsilon}}(\mathbf{p}_1)$ and $f_{\boldsymbol{\varepsilon}}(\mathbf{p}_2)$, i.e., by Lemma 2.3, $x \geq (1-\lambda)\sum_{k=0}^{n-1} a_k\lambda^k$. The same argument applies to y and z. □

Corollary 2.5. *We have*

$$f_{\boldsymbol{\varepsilon}}(H_0) = \begin{cases} x & < (1-\lambda)\left(\lambda^n + \sum_{k=0}^{n-1} a_k\lambda^k\right), \\ y & < (1-\lambda)\left(\lambda^n + \sum_{k=0}^{n-1} b_k\lambda^k\right), \\ z & < (1-\lambda)\left(\lambda^n + \sum_{k=0}^{n-1} c_k\lambda^k\right). \end{cases}$$

Proof. The argument is similar to the one in the proof of the previous lemma, so we skip it (note that $H_0 = \{(x,y,z) : x < 1-\lambda, y < 1-\lambda, z < 1-\lambda\}$). □

Lemma 2.1 cannot be reversed in the sense that any $f_{\boldsymbol{\varepsilon}}(H_0)$ is a hole, as we will see in Section 5. However, the following assertion shows that once we have one hole, we have infinitely many holes.

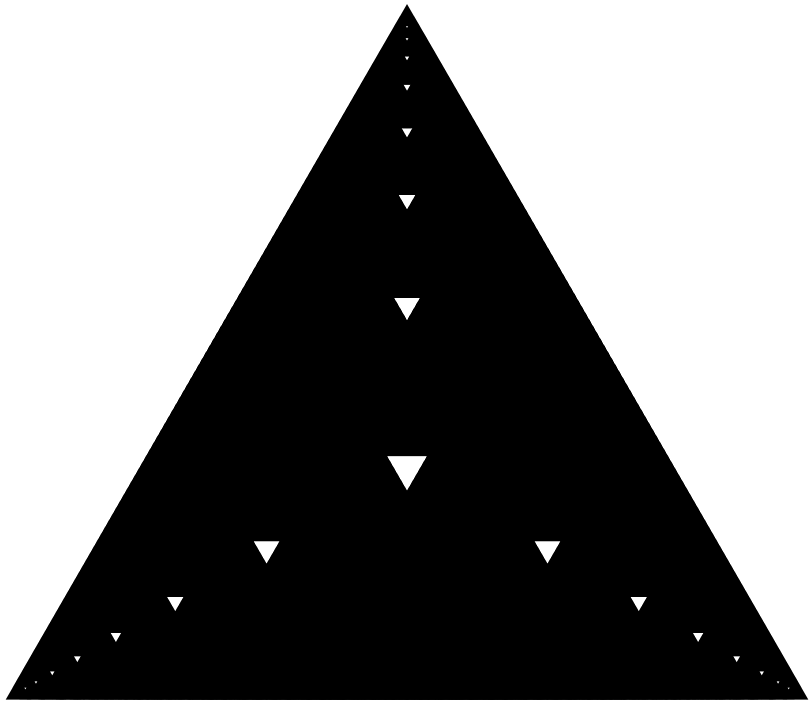


FIGURE 5. The invariant set for $\lambda = 0.65$.

Lemma 2.6. *For any $\lambda \in (1/2, 2/3)$ there is an infinite number of holes.*

Proof. We are going to show that $f_i^n(H_0)$ is always a hole for any $i = 0, 1, 2$ and any $n \geq 0$. In view of the symmetry, it suffices to show that $f_0^n(H_0)$ is a hole. By Corollary 2.5,

$$f_0^n(H_0) = \{(x, y, z) : x < 1 - \lambda^{n+1},\ y < \lambda^n(1 - \lambda),\ z < \lambda^n(1 - \lambda)\}. \tag{2.1}$$

Since the Δ_n are nested, it suffices to show that $f_0^n(H_0) \cap \Delta_{n+1} = \emptyset$. By Corollary 2.4, this means that the system of inequalities of the form

$$x \geq (1 - \lambda) \sum_0^n a_k \lambda^k,\ y \geq (1 - \lambda) \sum_0^n b_k \lambda^k,\ z \geq (1 - \lambda) \sum_0^n c_k \lambda^k \tag{2.2}$$

never occurs for $(x, y, z) \in f_0^n(H_0)$. Indeed, if it did, then by (2.1), we would have $b_j = c_j = 0$ for $0 \leq j \leq n$, whence $a_0 = \cdots = a_n = 1$, and by (2.2), $x \geq (1 - \lambda)(1 + \lambda + \cdots + \lambda^n) = 1 - \lambda^{n+1}$, which contradicts (2.1). $\square$

We call any hole of the form $f_i^n(H_0)$ a *radial hole*.

Proposition 2.7. *Let $\lambda_* \approx 0.6478$ be the appropriate root of*

$$x^3 - x^2 + x = \frac{1}{2}.$$

Then $\mathcal{S}_\lambda$ has a nonempty interior if $\lambda \in [\lambda_*, 2/3)$ and moreover, each hole is radial—see Figure 5.

Proof. We[3] need to show that for each $n \ge 0$,

$$F_n := \Delta_n \setminus \Delta_{n+1} \subset \bigcup_{k=0}^{n} \bigcup_i f_i^k(H_0)$$

(within this proof i always runs from 0 to 2). This is obvious for $n = 0, 1$, in view of the fact that H_0 and the $f_i(H_0)$ are always holes.

Thus, we have to show that $F_n \setminus F_{n-1}$ consists just of three holes for each $n \ge 3$. In view of the symmetry of our model, this is equivalent to the fact that

$$f_1 f_0^{n-1}(H_0) \cap F_n = \emptyset.$$

We have $F_n = \bigcap_i (\Delta_n \setminus f_i(\Delta_n))$. It suffices to show that $f_1 f_0^{n-1}(H_0) \subset f_0(\Delta_n)$. This in turn follows from the following relations:

(1) $f_1 f_0^{n-1}(H_0) \subset f_0(\Delta) \cap f_1(\Delta)$;
(2) $f_1 f_0^{n-1}(H_0) \cap f_0 f_1^{n-1}(H_0) = \emptyset$.

Let P be the vertex of H_0 with barycentric coordinates $(2\lambda - 1, 1 - \lambda, 1 - \lambda)$. Then (1) is effectively equivalent to $f_1 f_0^{n-1}(P) \in f_0(\Delta)$, which by Lemma 2.3, leads to $\lambda^{n+1} - \lambda^n + \lambda \ge \frac{1}{2}$. By monotonicity of the root of this polynomial with respect to n, the worst case scenario is $n = 2$, which is equivalent to $\lambda \ge \lambda_*$.

Let $Q = (1 - \lambda, 2\lambda - 1, 1 - \lambda)$. The condition (2) is equivalent to the fact that the x-coordinate of $f_1 f_0^{n-1}(P)$ is bigger than the x-coordinate of $f_0 f_1^{n-1}(Q)$, which, in view of Lemma 2.3, yields the inequality $\lambda(1 - 2\lambda^{n-1} + 2\lambda^n) > (1 - \lambda)(1 + \lambda^n)$ which is equivalent to

$$3\lambda^{n+1} - 3\lambda^n + 2\lambda > 1. \tag{2.3}$$

The worst case scenario is $n = 3$, where (2.3) is implied by $\lambda > 0.6421$, i.e., well within the range. □

Remark 2.8. As is easy to see, λ_* is the exact lower bound for the "purely radial" case, because if $\lambda < \lambda_*$, the set $f_1 f_0(H_0) \setminus f_0(\Delta)$ has an empty intersection with Δ_3 and hence is a hole. The details are left to the interested reader.

We finish this section by showing that the boundaries of $f_{\boldsymbol{\varepsilon}}(\Delta)$ do not contain holes.

Proposition 2.9. *For $\lambda \ge 1/2$*

$$\partial\Delta \subset \mathcal{S}_\lambda.$$

Consequently, for any $\boldsymbol{\varepsilon}$,

$$\partial f_{\boldsymbol{\varepsilon}}(\Delta) \subset \mathcal{S}_\lambda.$$

Proof. In barycentric coordinates, $\partial\Delta = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$. In view of the symmetry, it suffices to show that $K = \{z = 0\} \subset \mathcal{S}_\lambda$. Any point of K is of the form $(x, 1 - x, 0)$ with $x \in [0, 1]$. Now our claim follows from Lemma 2.3 and the fact that every $x \in [0, 1]$ has the greedy expansion in decreasing powers of λ, i.e., $x = (1 - \lambda) \sum_1^\infty a_k \lambda^k$. For y we put $b_k = 1 - a_k$.

For the second statement, since $\mathcal{S}_\lambda$ is invariant, $f_i(\mathcal{S}_\lambda) \subset \mathcal{S}_\lambda$, whence $f_{\boldsymbol{\varepsilon}}(\mathcal{S}_\lambda) \subset \mathcal{S}_\lambda$ for each $\boldsymbol{\varepsilon}$. Now our claim follows from $\partial f_{\boldsymbol{\varepsilon}}(\Delta) = f_{\boldsymbol{\varepsilon}}(\partial\Delta)$, together with the first part. □

It follows from this proposition that $\dim_H(\mathcal{S}_\lambda) \ge 1$.

[3]We are indebted to B. Solomyak whose suggestions have helped us with the idea of this proof.

3. Golden Gaskets

Within this section, let λ be equal to the *multinacci number* ω_m, i.e., the unique positive root of

$$x^m + x^{m-1} + \cdots + x = 1, \quad m \geq 2.$$

For every m, $\omega_m \in (1/2, 1)$. In particular, ω_2 is the golden ratio, $\omega_2 = \frac{\sqrt{5}-1}{2} \approx 0.618$, $\omega_3 \approx 0.544$, etc. It is well known that $\omega_m \searrow 1/2$ as $m \to +\infty$. To simplify our notation, we simply write ω instead of ω_m within this section, as our arguments are universal.

We will show that $\mathcal{S}_\omega$ is totally self-similar (Theorem 3.3); in Section 5 the converse will be proved. The key technical assertion is

Proposition 3.1. *The set $f_{\boldsymbol{\varepsilon}}(H_0)$ is a hole for any $\boldsymbol{\varepsilon} \in \Sigma^n$.*

Proof. Let Δ_n be given by (1.2), and

$$H_n := \bigcup_{\boldsymbol{\varepsilon} \in \Sigma^n} f_{\boldsymbol{\varepsilon}}(H_0), \quad n \geq 1. \tag{3.1}$$

As in Lemma 2.6, we show that $H_n \cap \Delta_{n+1} = \emptyset$. By Corollaries 2.4 and 2.5, it suffices to show that the inequalities

$$\begin{aligned} \omega^n + \sum_0^{n-1} a_k \omega^k &> \sum_0^n \alpha_k \omega^k, \\ \omega^n + \sum_0^{n-1} b_k \omega^k &> \sum_0^n \beta_k \omega^k, \\ \omega^n + \sum_0^{n-1} c_k \omega^k &> \sum_0^n \gamma_k \omega^k \end{aligned} \tag{3.2}$$

never hold simultaneously, provided all the coefficients are 0's and 1's, and $a_k + b_k + c_k = \alpha_k + \beta_k + \gamma_k = 1$.

The key to our argument is the following separation result (we use the conventional notation here):

Theorem 3.2. *(P. Erdős, I. Joó, M. Joó* [5, Theorem 4]*) Let $\theta > 1$, and*

$$\ell(\theta) := \inf \left\{ |\rho| : \rho = \sum_{k=0}^n s_k \theta^k \neq 0, s_k \in \{0, \pm 1\}, \ n \geq 1 \right\}. \tag{3.3}$$

Then $\ell(\theta) = \theta^{-1}$ if θ^{-1} is a multinacci number.

From this lemma we easily deduce a claim about the sums in question. Indeed, put $\theta = \omega^{-1}$ and assume that $a_k \in \{0, 1\}, a'_k \in \{0, 1\}$ for $k = 0, 1, \dots, n$, and $\sum_0^n a_k \omega^k > \sum_0^n a'_k \omega^k$. Then

$$\sum_0^n (a_k - a'_k) \omega^k \geq \omega^{n+1} \tag{3.4}$$

(just put $s_k = a_{n-k} - a'_{n-k}$).

We use inequality (3.4) to improve the inequalities (3.2). Formally set $a_n = b_n = c_n = 1$ and include the ω^n term of the left hand side of the inequalities (3.2) with the summation. Then by (3.4),

$$\sum_0^n a_k\omega^k \geq \sum_0^n \alpha_k\omega^k + \omega^{n+1},$$
$$\sum_0^n b_k\omega^k \geq \sum_0^n \beta_k\omega^k + \omega^{n+1},$$
$$\sum_0^n c_k\omega^k \geq \sum_0^n \gamma_k\omega^k + \omega^{n+1},$$

which is equivalent to

$$\begin{aligned}(1-\omega)\omega^n + \sum_0^{n-1} a_k\omega^k &\geq \sum_0^n \alpha_k\omega^k,\\ (1-\omega)\omega^n + \sum_0^{n-1} b_k\omega^k &\geq \sum_0^n \beta_k\omega^k,\\ (1-\omega)\omega^n + \sum_0^{n-1} c_k\omega^k &\geq \sum_0^n \gamma_k\omega^k.\end{aligned} \tag{3.5}$$

By our assumption, just one of the values $\alpha_n, \beta_n, \gamma_n$ is equal to 1. Let it be α_n, say; then the inequalities (3.5) may be rewritten as follows:

$$\begin{aligned}\sum_0^{n-1} a_k\omega^k &\geq \sum_0^{n-1} \alpha_k\omega^k + \omega^{n+1},\\ (1-\omega)\omega^n + \sum_0^{n-1} b_k\omega^k &\geq \sum_0^{n-1} \beta_k\omega^k,\\ (1-\omega)\omega^n + \sum_0^{n-1} c_k\omega^k &\geq \sum_0^{n-1} \gamma_k\omega^k.\end{aligned}$$

It suffices to again apply (3.4) to improve the first inequality. As $\sum_{k=0}^{n-1} a_k\omega^k > \sum_{k=0}^{n-1} \alpha_k\omega^k$, we have $\sum_{k=0}^{n-1} a_k\omega^k - \sum_{k=0}^{n-1} \alpha_k\omega^k \geq \omega^n$, whence

$$\begin{aligned}\sum_0^{n-1} a_k\omega^k &\geq \sum_0^{n-1} \alpha_k\omega^k + \omega^{n},\\ (1-\omega)\omega^n + \sum_0^{n-1} b_k\omega^k &\geq \sum_0^{n-1} \beta_k\omega^k,\\ (1-\omega)\omega^n + \sum_0^{n-1} c_k\omega^k &\geq \sum_0^{n-1} \gamma_k\omega^k.\end{aligned}$$

Summing up the left and right hand sides, we obtain, in view of $a_k + b_k + c_k = \alpha_k + \beta_k + \gamma_k = 1$,

$$2(1-\omega)\omega^n + \sum_0^{n-1} \omega^k \ge \omega^n + \sum_0^{n-1} \omega^k,$$

which implies $\omega \le 1/2$, a contradiction. □

This claim almost immediately yields the total self-similarity of the invariant set $\mathcal{S}_\omega$:

Theorem 3.3. *The set $\mathcal{S}_\omega$ is totally self-similar in the sense of Definition 1.2, i.e.,*

$$f_{\boldsymbol{\varepsilon}}(\mathcal{S}_\omega) = f_{\boldsymbol{\varepsilon}}(\Delta) \cap \mathcal{S}_\omega \quad \text{for any } \boldsymbol{\varepsilon} \in \Sigma^n.$$

Proof. Let H_n be defined by (3.1). Since $H_{n+k} = \bigcup_{\boldsymbol{\varepsilon}\in\Sigma^n} f_{\boldsymbol{\varepsilon}}(H_k)$, we have $f_{\boldsymbol{\varepsilon}}(H_k) \subset H_{n+k}$. Furthermore, $f_{\boldsymbol{\varepsilon}}(H_{k+1}) \subset f_{\boldsymbol{\varepsilon}}(\Delta)$, whence $f_{\boldsymbol{\varepsilon}}(H_k) \subset H_{n+k} \cap f_{\boldsymbol{\varepsilon}}(\Delta)$. On the other hand, by Proposition 3.1, either $f_{\boldsymbol{\varepsilon}}(H_0) \cap f_{\boldsymbol{\varepsilon}'}(H_0) = \emptyset$ or $f_{\boldsymbol{\varepsilon}}(H_0) = f_{\boldsymbol{\varepsilon}'}(H_0)$ for $\boldsymbol{\varepsilon} \in \Sigma^{n+k}$. Hence the elements of H_{n+k} are disjoint, and we have

$$f_{\boldsymbol{\varepsilon}}(H_k) = f_{\boldsymbol{\varepsilon}}(\Delta) \cap H_{n+k}.$$

Since we have proved in Proposition 3.1 that $H_{n+k} \cap \Delta_{n+k-1} = \emptyset$,

$$f_{\boldsymbol{\varepsilon}}(\Delta_k) = f_{\boldsymbol{\varepsilon}}(\Delta) \cap \Delta_{n+k-1}.$$

The claim now follows from the fact that $\Delta_k \to \mathcal{S}_\omega$ in the Hausdorff metric and from $f_{\boldsymbol{\varepsilon}}$ being continuous. □

4. Dimensions

Within this section we continue to assume $\lambda = \omega_m$ for some $m \ge 2$. From Proposition 3.1 it is easy to show that $\mathcal{S}_{\omega_m}$ is nowhere dense. We prove more than that:

Proposition 4.1. *The two-dimensional Lebesgue measure of $\mathcal{S}_{\omega_m}$ is zero.*

Proof. Our proof is based on Theorem 3.3. Note first that for any measure ν (finite or not),

$$\begin{aligned}\nu(\Delta) = {} & \nu(f_0(\Delta) \cup f_1(\Delta) \cup f_2(\Delta) \cup H_0) \\ & - \nu(f_0(\Delta) \cap f_1(\Delta)) - \nu(f_0(\Delta) \cap f_2(\Delta)) - \nu(f_1(\Delta) \cap f_2(\Delta)),\end{aligned}$$

whence by Theorem 3.3,

$$\begin{aligned}\nu\mathcal{S}_{\omega_m} = {} & \nu(f_0(\mathcal{S}_{\omega_m}) \cup f_1(\mathcal{S}_{\omega_m}) \cup f_2(\mathcal{S}_{\omega_m})) \\ & - \nu(f_0(\mathcal{S}_{\omega_m}) \cap f_1(\mathcal{S}_{\omega_m})) - \nu(f_0(\mathcal{S}_{\omega_m}) \cap f_2(\mathcal{S}_{\omega_m})) - \nu(f_1(\mathcal{S}_{\omega_m}) \cap f_2(\mathcal{S}_{\omega_m}))\end{aligned} \tag{4.1}$$

(because $H_0 \cap \mathcal{S}_{\omega_m} = \emptyset$). The central point of the proof is that there exists a simple expression for $f_i(\mathcal{S}_{\omega_m}) \cap f_j(\mathcal{S}_{\omega_m})$ for $i \ne j$. Namely,

$$f_i(\mathcal{S}_{\omega_m}) \cap f_j(\mathcal{S}_{\omega_m}) = f_i f_j^m(\mathcal{S}_{\omega_m}). \tag{4.2}$$

To prove this, note first that in view of Theorem 3.3, it suffices to show that

$$f_i(\Delta) \cap f_j(\Delta) = f_i f_j^m(\Delta). \tag{4.3}$$

Moreover, because of the symmetry of our model, in fact, we need to prove only that $f_0(\Delta) \cap f_1(\Delta) = f_0 f_1^m(\Delta)$. This in turn follows from Corollary 2.4:

$$f_0(\Delta) \cap f_1(\Delta) = \{(x, y, z) : x \ge 1 - \omega_m, y \ge 1 - \omega_m\}$$

and

$$f_0 f_1^m(\Delta) = \{(x, y, z) : x \ge 1 - \omega_m, y \ge (1 - \omega_m)(\omega_m + \cdots + \omega_m^m) = 1 - \omega_m\}.$$

The relation (4.2) is thus proved. Hence (4.1) can be rewritten as follows:

$$\begin{aligned}\nu\mathcal{S}_{\omega_m} &= \nu(f_0(\mathcal{S}_{\omega_m}) \cup f_1(\mathcal{S}_{\omega_m})) \cup f_2(\mathcal{S}_{\omega_m})) \\ &\quad - \nu(f_0 f_1^m(\mathcal{S}_{\omega_m})) - \nu(f_0 f_2^m(\mathcal{S}_{\omega_m})) - \nu(f_1 f_2^m(\mathcal{S}_{\omega_m})).\end{aligned} \tag{4.4}$$

Finally, let $\nu = \mu$, the two-dimensional Lebesgue measure scaled in such a way that $\mu(\Delta) = 1$. In view of the f_i being affine contractions with the same contraction ratio ω_m and by (4.4),

$$\mu(\mathcal{S}_{\omega_m}) = 3\omega_m^2 \mu(\mathcal{S}_{\omega_m}) - 3\omega_m^{2(m+1)} \mu(\mathcal{S}_{\omega_m}),$$

whence,

$$(1 - 3\omega_m^2 + 3\omega_m^{2(m+1)})\mu(\mathcal{S}_{\omega_m}) = 0. \tag{4.5}$$

It suffices to show that $1 - 3\omega_m^2 + 3\omega_m^{2(m+1)} \neq 0$. For $m = 2$, in view of $\omega_2^2 = 1 - \omega_2$, this follows from $1 - 3\omega_2^2 + 3\omega_2^6 = \omega_2^8 > 0$; for $m \geq 3$, we have $1 - 3\omega_m^2 + 3\omega_m^{2(m+1)} > 1 - 3\omega_m^2 > 0$, because $\omega_m \leq \omega_3 < 0.544 < 1/\sqrt{3}$.

Thus, by (4.5), $\mu(\mathcal{S}_{\omega_m}) = 0$. □

Remark 4.2. The only fact specific to the multinacci numbers that we used in this proof is the relation (4.3). It is easy to show that, conversely, this relation implies $\lambda = \omega$. We leave the details to the reader.

We do not know whether the Lebesgue measure of $\mathcal{S}_\lambda$ is zero if $\lambda < \omega_2$ (this is what the numerics might suggest), but a weaker result is almost immediate (NB: $\omega_2 > 1/\sqrt{3} > \omega_3$):

Proposition 4.3. *For any $\lambda < 1/\sqrt{3}$ the invariant set $\mathcal{S}_\lambda$ has zero Lebesgue measure.*

Proof. Since $\mathcal{S}_\lambda = f_0(\mathcal{S}_\lambda) \cup f_1(\mathcal{S}_\lambda) \cup f_2(\mathcal{S}_\lambda)$ it follows that

$$\mu(\mathcal{S}_\lambda) \leq 3\lambda^2 \mu(\mathcal{S}_\lambda).$$

As $\mathcal{S}_\lambda$ is bounded, we know that $\mu(\mathcal{S}_\lambda) < \infty$, so that either $\mu(\mathcal{S}_\lambda) = 0$ or $1 \leq 3\lambda^2$ as required. □

Return to the case $\lambda = \omega_m$. As $\mathcal{S}_{\omega_m}$ has zero Lebesgue measure, it is natural to ask what its Hausdorff dimension is. Let $\mathcal{H}^s$ denote the s-dimensional *Hausdorff measure*. As is well known,

$$\mathcal{H}^s(\lambda B + \mathbf{x}) = \lambda^s \mathcal{H}^s(B) \tag{4.6}$$

for any Borel set B, any vector $\mathbf{x}$ and any $\lambda > 0$. Let us compute $\mathcal{H}^s(\mathcal{S}_{\omega_m})$. By (4.4) and (4.6) with $\nu = \mathcal{H}^s$,

$$\mathcal{H}^s(\mathcal{S}_{\omega_m}) = 3\omega_m^s \mathcal{H}^s(\mathcal{S}_{\omega_m}) - 3\omega_m^{s(m+1)} \mathcal{H}^s(\mathcal{S}_{\omega_m}).$$

We see that unless

$$1 - 3\omega_m^s + 3\omega_m^{s(m+1)} = 0, \tag{4.7}$$

the s-Hausdorff measure of the invariant set is either 0 or $+\infty$. Recall that the value of d which separates 0 from $+\infty$ is called the *Hausdorff dimension* of a Borel set E (notation: $\dim_H(E)$). This argument relies on the invariant set having non-zero measure in the appropriate dimension, which we do not know, so in fact only amounts to a heuristic argument suggesting

Theorem 4.4. *The Hausdorff dimension of the invariant set $\mathcal{S}_{\omega_m}$ equals its box-counting dimension and is given by*

$$\dim_H(\mathcal{S}_{\omega_m}) = \dim_B(\mathcal{S}_{\omega_m}) = \frac{\log \tau_m}{\log \omega_m},$$

where τ_m is the largest root of the polynomial $3z^{m+1} - 3z + 1$.

The fact that the Hausdorff dimension coincides with the box-counting dimension for the attractor of a finite IFS is universal [7]. A rigorous proof of the formula for $\dim_H(\mathcal{S}_{\omega_m})$ is given in the appendix. It amounts to showing that the invariant set *essentially* coincides with the invariant set of a countably infinite IFS which satisfies the OSC.

Remark 4.5. The case $m = 2$ (the golden ratio) is especially nice as here

$$\tau_2 = \frac{2}{\sqrt{3}}\cos(7\pi/18).$$

The authors are grateful to H. Khudaverdyan for pointing this out. Note also that there cannot be such a nice formula for $m \geq 3$, because, as is easy to show, the Galois group of the extension $\mathbb{Q}(\tau_m)$ with $m \geq 3$ is symmetric.

Remark 4.6. Let us also mention that the set of holes, $\Delta \setminus \mathcal{S}_{\omega_2}$, can be identified with the Cayley graph of the semigroup

$$\Gamma := \{0, 1, 2 \mid 100 = 011, 200 = 022, 211 = 122\},$$

namely, $f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}(H_0)$ is identified with the equivalence class of the word $\varepsilon_0 \dots \varepsilon_{n-1}$. The relations $ij^2 = ji^2$, $i \neq j$ in Γ correspond to the relations $f_i f_j^2 = f_j f_i^2$, $i \neq j$.

Thus, $\Delta \setminus \mathcal{S}_{\omega_2}$ may be regarded as a generalization of the *Fibonacci graph*—the Cayley graph of the semigroup $\{0, 1 \mid 100 = 011\}$ introduced in [1] and studied in detail in [16].

Let u_n stand for the cardinality of level n of Γ (= the number of holes of the n^{th} layer). As is easy to see, $u_0 = 1, u_1 = 3, u_2 = 9$ and

$$u_{n+3} = 3u_{n+2} - 3u_n,$$

whence the rate of growth of Γ, $\lim_n \sqrt[n]{u_n}$, is equal to τ_2^{-1}. This immediately yields another proof that the box-counting dimension of $\mathcal{S}_{\omega_2}$ is equal to its Hausdorff dimension. The analogous results hold for $\lambda = \omega_m$ for any $m \geq 2$. We leave the details to the reader.

m	ω_m	$\dim_H(\mathcal{S}_{\omega_m})$
2	0.61803	1.93063
3	0.54369	1.73219
4	0.51879	1.65411
5	0.50866	1.61900
6	0.50414	1.60201
7	0.50202	1.59356
8	0.50099	1.58930
9	0.50049	1.58715
...	...	...
∞	$1/2$	$\log 3/\log 2$

TABLE 4.1. Hausdorff dimension of $\mathcal{S}_{\omega_m}$.

Remark 4.7. Recall that $\log 3/\log 2$ is the Hausdorff dimension of the Sierpiński sieve. From Theorem 4.4 it follows that $\dim_H(\mathcal{S}_{\omega_m}) \to \log 3/\log 2$ as $m \to +\infty$ (see also Table 4.1). Thus, although the Hausdorff dimension does not have to be continuous, in our case it is continuous as $m \to \infty$.

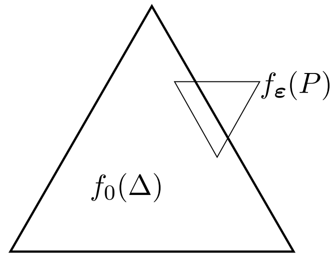


FIGURE 6. The pattern that always occurs unless $\lambda = \omega_m$

5. THE CONVERSE AND A NUMBER-THEORETIC APPLICATION

The aim of this section is to show that Theorem 3.3 can be reversed, i.e., the choice of multinacci numbers was not accidental. We are going to need some facts about λ-*expansions* of $x = 1$.

Note first that for every $\lambda \in (1/2, 1)$ there always exists a sequence $(a_k)_1^\infty$ (called a λ-*expansion*) that satisfies

$$1 = \sum_{k=1}^{\infty} a_k \lambda^k.$$

The reason why there always some λ-expansion available is because one can always take the *greedy expansion* of 1, namely, $a_k = [\lambda^{-1} T_\lambda^{k-1}(1)]$, where $[\cdot]$ stands for the integral part, and $T_\lambda(x) = x/\lambda - [x/\lambda]$ (see, e.g., [13]). We always assume in this preamble that $\mathbf{a} = (a_k)_1^\infty$ is the greedy λ-expansion of 1.

There is a convention in this theory that if the greedy expansion is of the form $(a_1, \dots, a_N, 0, 0, \dots)$, then it is replaced by $(a_1, \dots, a_N - 1)^\infty$ (this clearly does not change the value). For instance, the greedy expansion of 1 for $\lambda = \omega_2$ is $101010\dots$, and more generally, if $\lambda = \omega_m$, then $\mathbf{a} = (1^{m-1}0)^\infty$.

Remark 5.1. As is well known [13],

$$\sum_{k=n+1}^{\infty} a_k \lambda^k \le \lambda^n$$

for any $n \ge 0$, and the equality holds only if $\mathbf{a}$ is purely periodic, and $a_{n+j} \equiv a_j$ for each $j \ge 1$.

Lemma 5.2. *Unless λ is a multinacci number, there is always an n such that $a_n = 0,\ a_{n+1} = 1$ and $\sum_{k=n+1}^{\infty} a_k \lambda^k < \lambda^n$.*

Proof. It follows from Remark 5.1 that unless each 0 in $\mathbf{a}$ is followed by the string of L 1's for some $L \ge 1$ (which is exactly multinacci), the condition in question is always satisfied. ☐

Theorem 5.3. *If, for some $\lambda \in (1/2, 2/3)$, the invariant set $\mathcal{S}_\lambda$ is totally self-similar, then $\lambda = \omega_m$ for some $m \ge 2$.*

Proof. Assume λ is such that $\mathcal{S}_\lambda$ is totally self-similar. By definition of total self-similarity, $f_{\varepsilon}(H_0) \cap S_\lambda = \emptyset$ for any ε, i.e., the claim of Proposition 3.1 must be true. Therefore, it would be impossible that, say, $f_0(\Delta)$ had a "proper" intersection with $f_{\varepsilon}(H_0)$ for some ε (see Figure 6)—should this occur, a part of $\partial f_{\varepsilon}(\Delta)$ would have a hole, whence $\partial\Delta \not\subset \mathcal{S}_\lambda$, which contradicts Proposition 2.9.

Let us make the necessary computations. Put, as above, $P = (2\lambda - 1, 1 - \lambda, 1 - \lambda)$; then $f_{\varepsilon}(P)$ has the x-coordinate equal to $(2\lambda - 1)\lambda^n + (1 - \lambda) \sum_0^{n-1} a_k \lambda^k$ (just apply Lemma 2.3). Assume we have a situation exactly like in Figure 6. As is easy to see, $f_0(\Delta) = \{x \ge 1 - \lambda\}$, this x-coordinate

must be less than $1-\lambda$, whereas the x-coordinate of the side that bounds $f_{\boldsymbol{\varepsilon}}(H_0)$ must be less than $1-\lambda$. Thus,

$$\frac{2\lambda-1}{1-\lambda}\,\lambda^n < 1-\sum_1^{n-1} a_k\lambda^k < \lambda^n \tag{5.1}$$

(the sum begins at $k=1$, because obviously a_k must equal 0). Thus, we only need to show that if $\lambda\in(1/2,2/3)$ and not multinacci, then there always exists a 0-1 word $(a_1\dots a_{n-1})$ such that (5.1) holds.

Assume first that $1/2<\lambda<\omega_2$ and not a multinacci number. Let $\mathbf{a}$ be the greedy λ-expansion of 1; then by Lemma 5.2, there exists $n\ge 1$ such that $a_n=0, a_{n+1}=1$, and $1-\sum_0^{n-1}a_k\lambda^k = \sum_{n+1}^\infty a_k\lambda^k<\lambda^n$.

Consider the left hand side inequality in (5.1). Since $a_{n+1}=1$, we have

$$\sum_{n+1}^\infty a_k\lambda^k \ge \lambda^{n+1} > \frac{2\lambda-1}{1-\lambda}\,\lambda^n,$$

as $\lambda<\omega_2$ is equivalent to $\lambda^2+\lambda<1$, which implies $\lambda>(2\lambda-1)/(1-\lambda)$.

Assume now $\lambda>\omega_2$ (recall that there are no multinacci numbers here). Put $n=2$ and $a_1=1$. Then (5.1) turns into

$$\frac{2\lambda-1}{1-\lambda}\,\lambda^2<1-\lambda<\lambda^2,$$

which holds for $\lambda\in(\omega_2,\lambda_*)$, where λ_* is as in Proposition 2.7, i.e., the root of $2x^3-2x^2+2x-1=0$.

Thus, it suffices to consider $\lambda\in[\lambda_*,2/3)$. By Proposition 2.7, there are no holes in $f_0(\Delta)\cap f_1(\Delta)$ at all, which means that $\mathcal{S}_\lambda$ cannot be totally self-similar. The theorem is proved. ☐

Remark 5.4. Figure 7 shows consequences of $\mathcal{S}_\lambda$ being not totally self-similar. We see that the whole local structure gets destroyed.

Theorem 5.3 has a surprising number-theoretic application (recall the definition of $\ell(\theta)$ is given in Theorem 3.2):

Corollary 5.5. *Let $\theta\in(3/2,2)$. Then either θ^{-1} is multinacci or*

$$\ell(\theta)\le\frac{2}{2+\theta}<\frac{1}{\theta}. \tag{5.2}$$

Proof. Assume $\lambda=\theta^{-1}\ne\omega_m$ for any $m\ge 2$. From Theorem 5.3 it follows that our method of proving Proposition 3.1 simply would not work if λ was not a multinacci number. Recall that our proof was based on Theorem 3.2 which must consequently be wrong if λ is not multinacci.

Moreover, with $\kappa:=\theta\ell(\theta)$ and by the same chain of arguments as in the proof of Proposition 3.1, we come at the end to the inequality

$$2(1-\kappa\lambda)\ge\kappa$$

(in the original proof we had it with $\kappa=1$) which is equivalent to $\ell(\theta)\le\frac{2}{2+\theta}$. Thus, if this inequality is *not* satisfied, then the system of inequalities (3.2) does not hold either, which leads to the conclusion of Proposition 3.1 and consequently yields Theorem 3.3—a contradiction with Theorem 5.3. ☐

Remark 5.6. As is well known since the pioneering work [8], if θ is a *Pisot number* (an algebraic integer >1 whose Galois conjugates are all less than 1 in modulus), then $\ell(\theta)>0$ (note that the ω_m^{-1} are known to be Pisot). Furthermore, if θ is not an algebraic number satisfying an algebraic equation

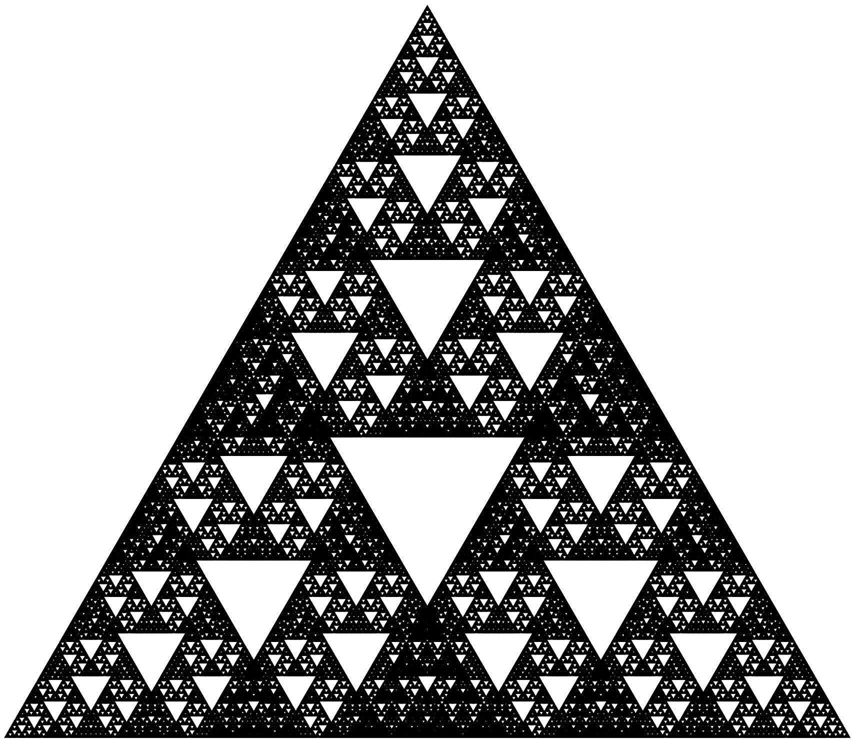


FIGURE 7. The invariant set for $\lambda = 0.59$. Observe that the holes up to the second "layer" seem to be intact, but start to "deteriorate" starting with the third "layer".

with coefficients $0, \pm 1$, then by the pigeonhole principle, $\ell(\theta) = 0$. There is a famous conjecture that this is also true for all algebraic non-Pisot numbers.

Thus (modulo this conjecture), effectively, the result of Corollary 5.5 is of interest if and only if θ is a Pisot number. The restriction $\theta > 3/2$ then is not really important, because in fact, there are only four Pisot numbers below $3/2$, namely, the appropriate roots of $x^3 = x + 1$ (the smallest Pisot number), $x^4 = x^3 + 1$, $x^5 - x^4 - x^3 + x^2 = 1$ and $x^3 = x^2 + 1$.[4] The respective values of $\ell(\theta)$ for these four numbers are approximately as follows: $0.06, 0.009, 0.002, 0.15$ (see [3]), i.e., significantly less than the estimate (5.2).

Thus, we have proved

Proposition 5.7. *For each Pisot number $\theta \in (1,2)$ that does not satisfy $x^m = x^{m-1} + x^{m-1} + \cdots + x + 1$ for some $m \geq 2$,*

$$\ell(\theta) \leq \frac{2}{2+\theta}.$$

For the history of the problem and the tables of $\ell(\theta)$ for some Pisot numbers θ see [3].

Remark 5.8. We are grateful to K. Hare who has indicated the paper [18] in which it is shown that $l(q) < 2/5$ for $q \in (1,2)$ and q^{-1} not multinacci. This is stronger than (5.2) but the proof in [18] is completely different, rather long and technical, so we think our result is worth mentioning.

[4]In fact, there is just a finite number of Pisot numbers below $\frac{1+\sqrt{5}}{2}$, and they all are known [2].

6. HIGHER-DIMENSIONAL ANALOGUES

The family of IFSs we have been considering consists of three contractions in the plane, with respective fixed points at the vertices of a regular 3-simplex in $\mathbb{R}^2$. In $\mathbb{R}^d$ it is natural to consider $d+1$ linear contractions with fixed points at the vertices of the $d+1$-simplex:

$$f_j(\mathbf{x}) = \lambda\mathbf{x} + (1-\lambda)\mathbf{p}_j, \quad (j = 0, \dots, d).$$

For example, when $d = 3$ the four maps are contractions towards the vertices of a regular tetrahedron in $\mathbb{R}^3$.

Using the analogous barycentric coordinate system (x_j is the distance to the j^{th} $(d-1)$-dimensional face of the simplex), the maps $f_0, \dots, f_d$ are given by matrices analogous to those in Lemma 1.3. The algebra of these maps is directly analogous to the family of three maps we have considered so far. The proofs of the following results are left as exercises (most are extensions of corresponding results earlier in the paper).

(1) If $\lambda \in [\frac{d}{d+1}, 1)$, then $\mathcal{S}_\lambda = \Delta$, so there are no holes in the attractor.

(2) If $\lambda \le 1/2$ the IFS satisfies the Open Set Condition, and the invariant set is self-similar with Hausdorff dimension

$$\dim_H(\mathcal{S}_\lambda) = \frac{\log(d+1)}{-\log\lambda}.$$

(3) Since the $(d+1)$-simplex contains the d-simplex at each of its faces, for any fixed λ we have $\mathcal{S}_\lambda(d+1) \supset \mathcal{S}_\lambda(d)$ and consequently,

$$\dim_H(\mathcal{S}_\lambda(d+1)) \ge \dim_H(\mathcal{S}_\lambda(d)).$$

(4) If $\lambda = \omega_m$ (the multinacci number), then the invariant set is totally self-similar, and the dimension s satisfies

$$s = \frac{\log\tau_{m,d}}{\log\omega_m}$$

where $\tau_{m,d}$ is the largest root of $\frac{1}{2}d(d+1)t^{m+1} - (d+1)t + 1 = 0$. See Table 6.1 for some values. One can see from this that, for fixed m and large d, the Hausdorff dimension increases logarithmically in d.

(5) If $\lambda < (d+1)^{-1/d}$, then—similarly to Proposition 4.3—$\mathcal{S}_\lambda$ has zero d-dimensional Lebesgue measure, but we do not know what happens for $\lambda \in \left((d+1)^{-1/d}, \frac{d}{d+1}\right)$.

d	ω_2	ω_3	ω_4	ω_5	ω_6	...	1/2
2	1.93	1.73	1.65	1.62	1.60	...	1.583
3	2.61	2.23	2.10	2.05	2.02	...	1.999
4	3.13	2.61	2.45	2.38	2.35	...	2.322
5	3.54	2.92	2.72	2.65	2.62	...	2.585
6	3.89	3.18	2.96	2.88	2.84	...	2.807

TABLE 6.1. Hausdorff dimension of golden d-gaskets

7. Final remarks and open questions

(1) The fact that the triangle is equilateral in our model is unimportant. Indeed, given any three non-collinear points $\mathbf{p}'_0, \mathbf{p}'_1, \mathbf{p}'_2$ in the plane there is a (unique) affine map A that maps each $\mathbf{p}'_j$ to the corresponding $\mathbf{p}_j$ we have been using. For given λ let $\mathcal{S}'_\lambda$ be the invariant set of the IFS defined by (1.1) with the $\mathbf{p}'_j$ in place of the $\mathbf{p}_j$. Then it is clear that $\mathcal{S}_\lambda = A(\mathcal{S}'_\lambda)$. For a given value of λ all the invariant sets are therefore affinely equivalent, and in particular have the same Hausdorff dimension (when this is defined).

(2) The sequence of golden gaskets $\mathcal{S}_{\omega_m}$ provides confirmation of some observations regarding the dimension of fractal sets generated by IFSs where the Open Set Condition fails. In particular, a theorem of Falconer [6] states that given linear maps $T_1, \dots, T_k$ on $\mathbb{R}^n$ of norm less than $1/3$, there is a number δ such that the invariant set $F(a_1, \dots, a_k)$ of the IFS $\{T_1 + a_1, \dots, T_k + a_k\}$ has Hausdorff dimension δ for a.e. $(a_1, \dots, a_k) \in \mathbb{R}^{nk}$. In the case that the T_j are all the same similarity by a factor of λ, the dimension is given by $\delta = \delta(\lambda) = -\log k / \log \lambda$.

It has been pointed out [17] that the upper bound $1/3$ can be replaced by $1/2$, but that the theorem fails if the upper bound is replaced by $1/2 + \varepsilon$ for any $\varepsilon > 0$. This can also be seen from the golden gaskets $\mathcal{S}_{\omega_m}$: given $\varepsilon > 0$ there is an m such that $1/2 < \omega_m < 1/2 + \varepsilon$, and the dimension of the invariant set $\dim_H(\mathcal{S}_{\omega_m}) < \delta(\omega_m)$.

(3) If one endows each of the maps f_i with probability $1/3$, this yields a *probabilistic IFS*. Its general definition can be found, for example, in the survey [4]. Then $\mathcal{S}_\lambda$ becomes the support for the *invariant measure*; the question is, what can be said about its Hausdorff dimension? In particular, we conjecture that, similarly to the 1D case (see [1, 16]), it is strictly less than $\dim_H(\mathcal{S}_\lambda)$ for $\lambda = \omega_m$.

(4) The main problem remaining is to determine for which λ the attractor $\mathcal{S}_\lambda$ has positive Lebesgue measure and for which zero Lebesgue measure. The numerics suggests the following

Conjecture. (1) For each $\lambda \in (\omega_2, 2/3)$ the attractor $\mathcal{S}_\lambda$ has a nonempty interior (recall that we know this for $\lambda \in [\lambda_*, 2/3)$—Proposition 2.7).
(2) For each $\lambda \in (1/\sqrt{3}, \omega_2)$ it has an empty interior and possibly zero Lebesgue measure.

(5) The same range of problems can be considered for any collection of similitudes $f_j(\mathbf{x}) = \lambda\mathbf{x} + (1-\lambda)\mathbf{p}_j$ in $\mathbb{R}^d$, where the $\mathbf{p}_j$ are vertices of a (convex) polytope Π. For instance, are there any totally-self similar attractors if Π is not a simplex and the OSC fails? This question seems to be particularly interesting if $d = 2$ and Π is regular n-gon with $n \geq 5$.

Appendix

We now give a rigorous proof of Theorem 4.4 (repeated below for convenience), using the fact that the invariant set almost coincides with the invariant set for an infinite IFS which satisfies the open set condition, and relying on some results about such systems [12]. We begin with an elementary lemma. Recall that the multinacci number ω_m is the unique root of $t^{m+1} - 2t + 1$ lying in $(\frac{1}{2}, \frac{2}{3})$.

Lemma A.1. *For each integer $m \geq 2$, let $\tau_m \in (0, 1/2)$ be the smaller positive root of $3t^{m+1} - 3t + 1$, and $\sigma_m \in (0, 1)$ the smaller positive root of $2t^m - 3t + 1$. Then*

$$\tfrac{1}{3} < \tau_m < \sigma_m < \omega_m < \tfrac{2}{3}.$$

Consequently

$$\frac{\log \tau_m}{\log \omega_m} > \frac{\log \sigma_m}{\log \omega_m} > 1.$$

Proof. Let $p_m = 3t^{m+1} - 3t + 1$ and $q_m = 2t^m - 3t + 1$. Notice that the derivatives of p_m and q_m are monotonic on the interval $[0, 1]$, so that each have at most two roots on that interval. Note also that $p_m(\omega_m) < 0$ and $q_m(\omega_m) < 0$. Since $p_m(1) = 1$ and $q_m(1) = 0$ and $p_m(\frac{1}{3}) > 0$ and $q_m(\frac{1}{3}) > 0$ it follows that $\frac{1}{3} < \tau_m < \omega_m$ and $\frac{1}{3} < \sigma_m < \omega_m$.

Finally, $p_m(\sigma_m) = 3\sigma_m\left(\frac{3\sigma_m-1}{2}\right) - 3\sigma_m + 1 = \frac{1}{2}(3\sigma_m - 1)(3\sigma_m - 2) < 0$, so that $\sigma_m > \tau_m$. □

To complete the picture (though we don't use this), if τ'_m and σ'_m are the other positive roots of p_m and q_m respectively, then $\omega_m < \tau'_m < \sigma'_m = 1$. Furthermore, $\lim_{m\to\infty} \tau_m = \lim_{m\to\infty} \sigma_m = \frac{1}{3}$.

Theorem 4.4. *The Hausdorff dimension of the invariant set $\mathcal{S}_{\omega_m}$ is given by*

$$\dim_H(\mathcal{S}_{\omega_m}) = \frac{\log \tau_m}{\log \omega_m},$$

where τ_m is defined in the lemma above.

Definition A.2. An alternative definition of $\mathcal{S}_\lambda$ is as follows (see, e.g., [4]): to any $\boldsymbol{\varepsilon} \in \Sigma^\infty$ there corresponds the unique point $\mathbf{x}_{\boldsymbol{\varepsilon}} = \lim_{n\to\infty} f_{\varepsilon_0} \dots f_{\varepsilon_n}(\mathbf{x}_0) \in \mathcal{S}_\lambda$. This limit does not depend on the choice of $\mathbf{x}_0$; we call $\boldsymbol{\varepsilon}$ an *address* of $\mathbf{x}_{\boldsymbol{\varepsilon}}$. Note that a given $\mathbf{x} \in \mathcal{S}_\lambda$ may have more than one address—see Proposition A.5 below.

Definition A.3. Let $\mathcal{U}_\lambda$ denote the *set of uniqueness*, i.e.,

$$\mathcal{U}_\lambda = \{\mathbf{x} \in \Delta \mid \exists!(\varepsilon_0, \varepsilon_1, \dots) : \mathbf{x} = \mathbf{x}_{\boldsymbol{\varepsilon}}\}.$$

In other words, $\mathcal{U}_\lambda$ is the set of points in $\mathcal{S}_\lambda$, each of which has a unique address. These sets seem to have an interesting structure for general λ's, and we plan to study them in subsequent papers. Note that in the one-dimensional case ($f_j(x) = \lambda x + (1-\lambda)j,\ j = 0, 1$) such sets have been studied in detail by P. Glendinning and the third author in [9].

In the course of the proof of this theorem, we also prove the following

Theorem A.4. *The set of uniqueness $\mathcal{U}_{\omega_m}$ is a self-similar set of Hausdorff dimension*

$$\dim_H(\mathcal{U}_{\omega_m}) = \frac{\log \sigma_m}{\log \omega_m},$$

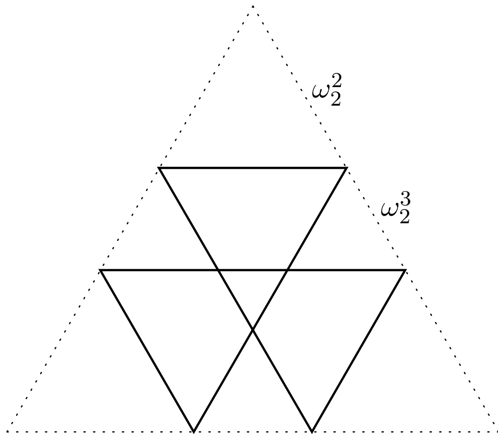
where σ_m is defined in Lemma A.1. In particular, $\sigma_2 = 1/2$.

Proof of both theorems. The proof proceeds by showing that there is another IFS (an infinite one) which does satisfy the OSC, and whose invariant set $\mathcal{A}_{\omega_m}$ satisfies $\mathcal{S}_{\omega_m} = \mathcal{A}_{\omega_m} \cup \mathcal{U}_{\omega_m}$, with $\dim_H(\mathcal{U}_{\omega_m}) < \dim_H(\mathcal{A}_{\omega_m})$. It then follows that $\dim_H(\mathcal{S}_{\omega_m}) = \dim_H(\mathcal{A}_{\omega_m})$, and the latter is given by a simple formula.

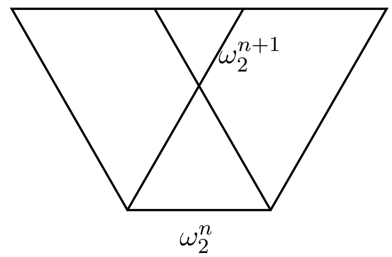
The proof for $m = 2$ differs in the details from that for $m > 2$ so we treat the cases separately. Note that within this appendix we assume the triangle Δ has unit side.

The case m= 2. Refer to Figure 8 for the geometry of this case. Begin by removing from the equilateral triangle Δ the (open) central hole H_0, the three (closed) triangles of side ω_2^2 that are the images of the three f_j^2 ($j = 0, 1, 2$), and three smaller triangles of side ω_2^3 that are the images of $f_i f_j^2 = f_j f_i^2$. This leaves three trapezia, whose union we denote T_1. See Figure 8 (a). For this part of the proof we write $F_k = f_i f_j^2$ (where i, j, k are distinct).

Each of the three trapezia is decomposed into the following sets: a hole (together forming H_1), an equilateral triangle of side ω_2^4, and two smaller trapezia—smaller by a factor of ω_2. The three equilateral triangles at this level are the images of $f_1 f_0 f_2^2 = f_1 F_1$ (for the lower left trapezium), $f_2 F_2$



(a) 3 trapezia



(b) decomposition of a trapezium at level n: 1 hole, 1 triangle and 2 (smaller) trapezia

FIGURE 8. Decomposing the golden gasket ($\lambda = \omega_2$)

(lower right) and f_0F_0 (upper trapezium). See Figure 8 (b). At the next level the equilateral triangles are the images of $f_jf_iF_i$ with $i \neq j$, and at the following $f_kf_jf_iF_i$ with $k \neq j$ and $j \neq i$.

This decomposition of the trapezia is now continued *ad infinitum*. At the n^{th} level there are $3 \cdot 2^{n-1}$ holes forming H_n, the same number of equilateral triangles that are images of similarities by ω_2^n and twice as many trapezia. Note that at each stage, the holes consist of those points with no preimage, the equilateral triangles of those points with two preimages and the trapezia of points with a unique preimage.

Let $\mathcal{A}_{\omega_2}$ be defined as the attractor corresponding to the equilateral triangles in the above construction; thus, it is the attractor for the infinite IFS with generators

$$\left\{f_j^2, F_j, f_jF_j, f_jf_iF_i, f_kf_jf_iF_i, \dots\right\}, \tag{A.1}$$

where the general term is of the form $f_{j_1}f_{j_2}\dots f_{j_n}F_{j_n}$ with adjacent j_k different from each other. Notice that this IFS satisfies the open set condition. In [12] a deep theory of *conformal IFS* (which our linear one certainly is) has been developed. From this theory it follows that, similarly to the finite IFSs, the Hausdorff dimension s of the invariant set $\mathcal{A}$ (henceforward we drop the subscript 'ω_2') equals its similarity dimension given by $\omega_2^s = \tau$, where in our case, τ is a solution of

$$1 = 3\tau^2 + 3\tau^3 + 3\tau^4 \sum_{n=0}^{\infty} 2^n \tau^n.$$

This equation has a unique positive solution and is equivalent to $(3\tau^3 - 3\tau + 1)(\tau + 1) = 0$ provided $\tau < 1/2$ (the radius of convergence of the above power series). Thus, τ is the solution of

$$3\tau^3 - 3\tau + 1 = 0,$$

with $\tau < 1/2$, in agreement with the value of the dimension of $\mathcal{S}$ given in the theorem.

Since this IFS is contained in the original IFS (generated by the f_i), so $\mathcal{A} \subset \mathcal{S}$.

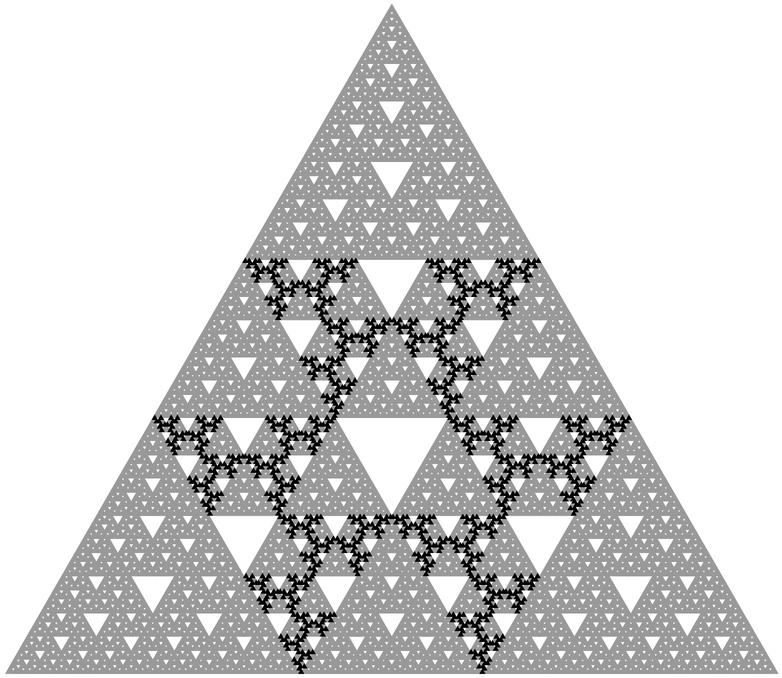


FIGURE 9. The fractal set $\mathcal{U}'_{\omega_2}$ superimposed on a grey $\mathcal{S}_{\omega_2}$.

Now let $\mathcal{U}'$ be the limit of the sequence of unions of trapezia defined by the above procedure: write $\mathcal{U}^{(n)}$ for the union of the $3 \cdot 2^{n-1}$ trapezia obtained at the n^{th} step, then $\mathcal{U}^{(n+1)} \subset \mathcal{U}^{(n)}$ and

$$\mathcal{U}' = \bigcap_{n>0} \mathcal{U}^{(n)}.$$

By construction, $\mathcal{U}'$ is a connected self-similar Cantor set, with dimension

$$\dim_H(\mathcal{U}') = -\frac{\log 2}{\log \omega_2}. \tag{A.2}$$

This follows from the standard arguments, since $\#\,\mathcal{U}^{(n)} \asymp 2^n$ and $\operatorname{diam} \mathcal{U}^{(n)} \asymp \omega_2^n$.

We claim that

$$\mathcal{U}' \cup \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2} f_k^{2n}(\mathcal{U}') = \mathcal{U}.$$

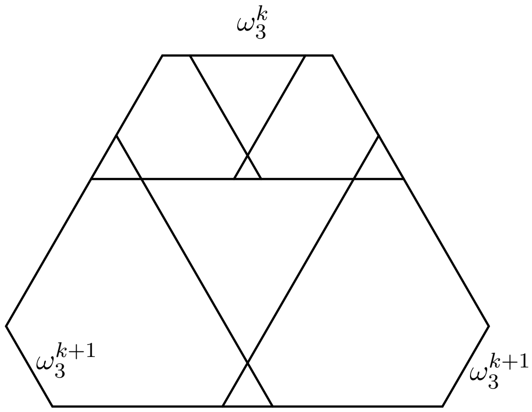
To see this, we turn to the addresses in the symbol space Σ. In view of the relation $f_i f_j^2 = f_j f_i^2$, each $\mathbf{x} \in \mathcal{S}$ that has multiple addresses, must have $\varepsilon_{j-1} \neq \varepsilon_j, \varepsilon_j = \varepsilon_{j+1}$ for some $j \geq 1$. By our construction, this union is the set of $\mathbf{x}$'s whose addresses can have equal symbols only at the beginning. Thus, it is indeed the set of uniqueness.

Since this is a countable union of sets of the same dimension, it follows that $\dim_H \mathcal{U} = \dim_H \mathcal{U}'$ (see, e.g., [7]). We claim that

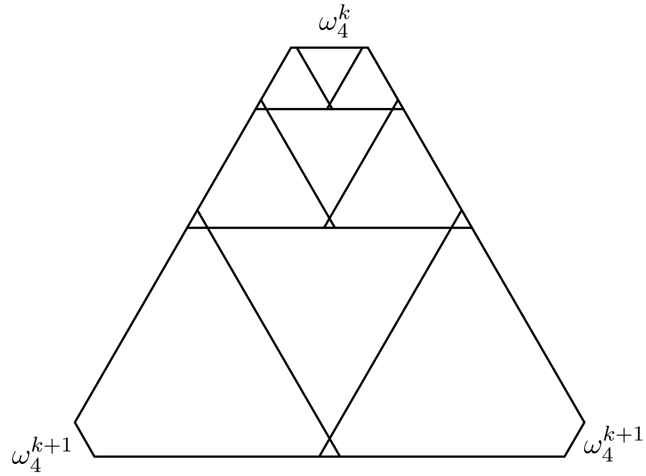
$$\mathcal{S} = \mathcal{A} \cup \mathcal{U} \tag{A.3}$$

with the union being disjoint. The result (for ω_2) then follows, since $\dim \mathcal{A} > \dim \mathcal{U}$.

To verify (A.3), we observe that from (A.1) it follows that $\mathbf{x} \in \mathcal{A}$ if and only if $\mathbf{x} = \mathbf{x}_{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon}$ for which there are two consecutive indices which coincide and do not coincide with the previous



(a) $m = 3$
2 holes, 4 triangles and 4 ω-hexagons



(b) $m = 4$
3 holes, 7 triangles and 6 ω-hexagons

FIGURE 10. Decomposing an ω-hexagon of size ω_m^k

one, i.e., $\mathcal{A} \cap \mathcal{U} = \emptyset$. Conversely, every point in $\mathcal{S}$ with more than one address lies in $\mathcal{A}$. Thus, apart from the three vertices of Δ, $\mathcal{S} \setminus \mathcal{A}$ consists of the points with a unique address, and expression (A.2) proves Theorem A.4 for the case $m = 2$.

The case $m \geq 3$. The overall argument is similar to that for $m = 2$, except that the trapezia are replaced by hexagons, and the recurrent structure is consequently different (more complicated); we only describe where the arguments differ.

We begin in the same way, by removing the central hole H_0, and decomposing the remainder into 3 small triangles of side ω_m^m at the vertices—the images of f_j^m, 3 smaller triangles of side ω_m^{m+1} on each side—images of $f_i f_j^m$, and 3 remaining hexagons (instead of trapezia).

These hexagons have sides of length ω_m^m, $(1 - \omega_m - \omega_m^m)$, ω_m^{m+1}, $(2 - 3\omega_m)$, ω_m^{m+1} and $(1 - \omega_m - \omega_m^m)$ (in cyclic order). We call hexagons similar to these, ω-hexagons, and this one in particular an ω-hexagon of *size* ω^m. Notice that these ω-hexagons have a single line of symmetry, and the *size* refers to the length of the smaller of the two sides that meet this line of symmetry.

Each ω-hexagon of size ω_m^k can be decomposed into: $(m-1)$ holes of various sizes down the line of symmetry; $(3m-5)$ equilateral triangles, 3 each of sizes $\omega_m^{k+2}, \omega_m^{k+3}, \dots, \omega_m^{k+m-1}$ and one of size ω_m^{k+m}; this leaves $2(m-1)$ ω-hexagons, 2 each of sizes $\omega_m^{k+1}, \omega_m^{k+2}, \dots, \omega_m^{k+m-1}$ (see Figure 10 for the cases $m = 3$ and 4).

In the same way as in the case $m = 2$, the equilateral triangles occurring in this decomposition are the images of the original triangle Δ under certain similarities arising in the IFS generated by $\{f_0, f_1, f_2\}$. This sub-IFS defines a countable IFS which satisfies the OSC, permitting us again to compute the dimension of the corresponding invariant set $\mathcal{A}_{\omega_m}$. We use generating functions to compute this dimension.

Each hexagon of size ω_m^k decomposes into 2 hexagons of sizes $\omega_m^{k+1}, \dots, \omega_m^{k+m-1}$. Thus, each hexagon of size ω_m^k arises from decomposing hexagons of sizes $\omega_m^{k-m+1} \dots \omega_m^{k-1}$. Let h_k be the number of hexagons of size ω_m^k that appear in the procedure. Then, $h_k = 0$ for $k < m$, $h_m = 3$ and

for $k > m$,

$$h_k = 2\left(h_{k-m+1} + \cdots + h_{k-1}\right).$$

Applying the usual generating function approach, let $Q = \sum_{k=1}^{\infty} h_k t^k$. Then

$$\begin{aligned} Q &= 3t^m + 2\sum_{k=m+1}^{\infty}\sum_{r=1}^{m-1} h_{k-r}t^k \\ &= 3t^m + 2\sum_{r=1}^{m-1} t^r \sum_{k=m+1}^{\infty} h_{k-r}t^{k-r} \\ &= 3t^m + 2Q\sum_{r=1}^{m-1} t^r. \end{aligned}$$

Finally, provided $|t| < r_m$ the radius of convergence of the power series,

$$Q = \frac{3t^m(1-t)}{1-3t+2t^m}.$$

Note from its definition in Lemma A.1 that $r_m = \sigma_m$. Now for the triangles: each ω-hexagon of size ω_m^k gives rise to 3 triangles of sizes $\omega_m^{k+2}, \ldots, \omega_m^{k+m-1}$ and one of size ω_m^{k+m}. Let there be p_k triangles of size ω_m^k. Then $p_k = 0$ for $k < m$, $p_m = p_{m+1} = 3$, and for $k > m+1$,

$$p_k = h_{k-m} + 3(h_{k-m+1} + \cdots + h_{k-2}).$$

Let $P = \sum_{k=0}^{\infty} p_k t^k$. Then

$$\begin{aligned} P &= 3t^m + 3t^{m+1} + t^m Q + 3\left(t^2 + \cdots + t^{m-1}\right)Q \\ &= 3t^m \frac{1-2t+t^{m+1}}{1-3t+2t^m}, \end{aligned}$$

again provided $|t| < \sigma_m$.

The formula for the Hausdorff dimension of the infinite IFS is just $s = \log\tau_m/\log\omega_m$, where by [12, Corollary 3.17], τ_m is the supremum of all x such that $\sum_k p_k x^k < 1$, i.e., the (unique) positive root of $\sum_k p_k x^k = 1$. In other words, it is the unique solution in $(0, \sigma_m)$ of

$$3\tau^m \frac{1-2\tau+\tau^{m+1}}{1-3\tau+\tau^m} = 1.$$

Rearranging this equation, one finds

$$(3\tau^{m+1} - 3\tau + 1)\mathfrak{C} = 0,$$

where $\mathfrak{C}$ is the polynomial $\mathfrak{C} = 1 + t + \cdots + t^m$, none of whose roots are positive. It follows that

$$\dim_H(\mathcal{A}_{\omega_m}) = \log\tau_m/\log\omega_m.$$

It remains to show that $\dim_H(\mathcal{A}_{\omega_m}) = \dim_H(\mathcal{S}_{\omega_m})$. The argument is similar to that for ω_2: it suffices to evaluate the growth of the number of hexagons, which follows from the generating function Q found above. Indeed, the growth of the coefficient is asymptotically $h_k \asymp \sigma_m^{-k}$ since σ_m is the smallest root of the denominator of Q (the radius of convergence mentioned above). Thus, by Lemma A.1,

$$\dim_H(\mathcal{U}_{\omega_m}) = \frac{\log\sigma_m}{\log\omega_m} < \frac{\log\tau_m}{\log\omega_m} = \dim(\mathcal{A}_{\omega_m}).$$

The argument showing that $\mathcal{U}_{\omega_m}$ is indeed the set of uniqueness, is analogous to the case $m = 2$, so we omit it. Theorems 4.4 and A.4 are now established. $\square$

Thus, we have shown that "almost every" point of $\mathcal{S}_{\omega_m}$ (in the sense of prevailing dimension) has at least *two* different addresses. It is easy to prove a stronger claim:

Proposition A.5. *Define $\mathcal{C}_\lambda$ as the set of points in $\mathcal{S}_\lambda$ with less than a continuum addresses, i.e.,*

$$\mathcal{C}_\lambda := \left\{ \mathbf{x} \in \mathcal{S}_\lambda : \mathrm{card}\{\boldsymbol{\varepsilon} \in \Sigma^\infty : \mathbf{x} = \mathbf{x}_{\boldsymbol{\varepsilon}}\} < 2^{\aleph_0} \right\}.$$

Then

$$\dim_H(\mathcal{C}_{\omega_m}) = \dim_H(\mathcal{U}_{\omega_m}) < \dim_H(\mathcal{S}_{\omega_m}).$$

Proof. Let $\mathbf{x} = \mathbf{x}_{\boldsymbol{\varepsilon}}$; if there exist an infinite number of k's such that $\varepsilon_k = i_k, \varepsilon_{k+1} = \cdots = \varepsilon_{k+m} = j_k$ with $i_k \neq j_k$, then, obviously, $\mathbf{x}$ has a continuum of addresses, because one can replace each $i_k j_k^m$ by $j_k i_k^m$ independently of the rest of the address.

Thus, $\mathbf{x} = \mathbf{x}_{\boldsymbol{\varepsilon}} \in \mathcal{C}_{\omega_m}$ only if the tail of $\boldsymbol{\varepsilon}$ is either j^∞ for some $j \in \Sigma$ (a countable set we discard) or a sequence $\boldsymbol{\varepsilon}'$ such that $\mathbf{x}_{\boldsymbol{\varepsilon}'} \in \mathcal{U}'_{\omega_m}$. Hence $\mathcal{C}_{\omega_m}$ contains a (countable) union of images of $\mathcal{U}'_{\omega_m}$, each having the same Hausdorff dimension, whence $\dim_H(\mathcal{C}_{\omega_m}) = \dim_H(\mathcal{U}_{\omega_m})$. ☐

We conjecture that the same claim is true for each $\lambda \in (1/2, 1)$. For the one-dimensional model this was shown by the third author [15]. Note that combinatorial questions of such a kind make sense for $\lambda \geq 2/3$ as well, since here the holes are unimportant.

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Department of Mathematics,
UMIST,
P.O. Box 88,
Manchester M60 1QD,
United Kingdom.

E-mails:
d.s.broomhead@umist.ac.uk
j.montaldi@umist.ac.uk
nikita.a.sidorov@umist.ac.uk